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Note

On the Solution of the Ambartsumian–Chandrasekhar Equation by Monotone Iteration Processes

I. INTRODUCTION

In the theory of radiative transfer in semi-infinite atmospheres an important part is played by the nonlinear integral equation [2, 6]

$$y(x) = 1 + y(x) \int_0^1 \frac{x}{x+t} y(t) f(t)\, dt, \tag{1}$$

where the characteristic function f is nonnegative and satisfies $\int_0^1 f(t)\, dt \leqslant \frac{1}{2}$. If we set $z(x) = y(x) f(x)$, (1) can be written in the form

$$z(x) = f(x) + z(x) \int_0^1 \frac{x}{x+t} z(t)\, dt$$

or, if operator notation is used, we can write

$$z = Tz = f + B(z, z). \tag{1'}$$

It has been shown [3, 4] that Eq. (1) has a unique solution z^* in the set

$$S = \{z \in K : \| z \| \leqslant \tfrac{1}{2} + \| f \|\},$$

which is mapped into itself by T. Here we have denoted by $\| \ \|$ the L_1 norm and by K the positive cone of the real vector space $L_1[0, 1]$, i.e. the set of functions of $L_1[0, 1]$ which are nonnegative almost everywhere on $[0, 1]$. The cone K induces on $L_1[0, 1]$ a partial ordering: u precedes v if and only if $v - u \in K$. The definition of monotone increasing (decreasing) sequence $\{u_n\}$ in $L_1[0, 1]$ follows in a natural way. In case of Eq. (1′) it can be shown that the sequence

$$u_0 = f, \qquad u_{n+1} = Tu_n \tag{2}$$

is monotone increasing and uniformly convergent to z^*.

Iterative procedures for solving Eq. (1) have been analyzed by several authors; see, for example, Stibbs and Weir [12], Moore [8], Noble [9], Rall [11]. Monotone iteration processes have been considered by Rall [10] and Casadei [5].

In Section II we shall establish a criterion for constructing a monotone decreasing sequence $\{v_n\}$ uniformly converging to z^* so that two-sided approximations to z^* are available, giving an estimate of the error at each stage of the process. Moreover, since the usual iteration process is often slowly convergent, a technique is described in Section III to accelerate the convergence of $\{u_n\}$ and $\{v_n\}$. A numerical application of the given method is presented in Section IV.

II. Construction of a Monotone Decreasing Sequence

The result given below is analogous to that obtained in [10] in case of isotropic scattering; an extension to a more general class of integral equations is described in [5].

THEOREM 1. *Let the characteristic function f satisfy the condition*

$$0 < M = \int_0^1 \frac{1}{1+t} f(t)\,dt < \tfrac{1}{4} \tag{3}$$

and denote by c a real number for which

$$[1-2M-(1-4M)^{1/2}]/2M \leqslant c \leqslant \min\{1/(2\,\|f\|),\,[1-2M+(1-4M)^{1/2}]/2M\}. \tag{4}$$

Let the function v be defined by $v(x) = (1+c)f(x)\ \forall\, x \in [0,1]$. *Then* $v \in S$ *and* $Tv - v \leqslant 0$.

Proof.

$$\begin{aligned}(Tv)(x) - v(x) &= f(x)\left\{(1+c)^2 \int_0^1 \frac{x}{x+t} f(t)\,dt - c\right\}\\ &\leqslant f(x)[(1+c)^2 M - c].\end{aligned}$$

By (3) the polynomial $[(1+c)^2 M - c]$ has two distinct positive roots and by (4) the last term written is nonpositive. Moreover, from $c \leqslant 1/(2\,\|f\|)$ it follows that $\|v\| \leqslant \|f\| + \frac{1}{2}$ and therefore $v \in S$. Q.E.D.

Remark 1. The interval defined by (4) is not empty since

$$[1-2M-(1-4M)^{1/2}]/2M < 1$$

and, a fortiori, $\leqslant 1/(2\,\|f\|)$.

Remark 2. The monotone decreasing sequence

$$v_0 = v, \qquad v_{n+1} = Tv_n \tag{5}$$

is uniformly convergent to z^* as can easily be seen following an argument analogous to that used in [4].

III. A Method of Speeding up the Convergence

In this section we prove two propositions that provide a method of speeding up the convergence of (2) and (5) respectively. The proofs are based on the following properties of the bilinear operator B. Let $u, v \in K$, $u \leqslant v$. It is easily seen that

(i) $B(u, u) \leqslant B(v, v)$;

(ii) if a is a nonnegative real number, then $(1 + a) B(v, v) - aB(u, u) \leqslant B((1 + a) v - au, (1 + a) v - au)$; and

(iii) if b, $0 \leqslant b \leqslant 1$, is a real number, then

$$B(bu + (1 - b) v, bu + (1 - b) v) \leqslant bB(u, u) + (1 - b) B(v, v).$$

The method described in Theorem 2 is an extension of that proposed by Albrecht [1] for solving certain systems of linear algebraic equations.

Theorem 2. *Let $u_0 \in S$, and let $u_1 = Tu_0$, $u_2 = Tu_1$ satisfy $0 \leqslant u_2 - u_1 = d$; let a be a positive real number such that*

$$a \leqslant (\tfrac{1}{2} + \|f\| - \|u_2\|)/\|d\|, \tag{6.1}$$

$$a(u_1 - u_0 - d) \leqslant d, \tag{6.2}$$

and put $u^* = u_2 + ad$. Then $u^* \in S$ and $u^* - Tu^* \leqslant 0$.

Proof. It is obviously true that $u^* \in S$ by virtue of (6.1). In addition, using property (ii) we have

$$\begin{aligned} u^* - Tu^* &= (1 + a) B(u_1, u_1) - aB(u_0, u_0) - B((1 + a) u_2 - au_1, \\ &\qquad\qquad (1 + a) u_2 - au_1) \\ &\leqslant B((1 + a) u_1 - au_0, (1 + a) u_1 - au_0) - B((1 + a) u_2 - au_1, \\ &\qquad\qquad (1 + a) u_2 - au_1). \end{aligned}$$

From (6.2) and (i) it follows that the last expression is nonpositive. Q.E.D.

Remark. The above method is independent of condition (3) and therefore can be always used to speed up the convergence of (2).

To accelerate the convergence of sequence (5) we combine elements of both sequences $\{u_n\}$ and $\{v_n\}$. The technique here has a direct analogy to that used in [13].

Theorem 3. *Let $u_0, v_0 \in S$, $u_0 < v_0$, and let $u_1 = Tu_0$, $v_1 = Tv_0$ satisfy $u_0 - u_1 \leqslant 0$, $v_1 - v_0 \leqslant 0$; moreover let b, $0 \leqslant b < 1$ be a real number for which*

$$b(u_1 - u_0 + v_0 - v_1) \leqslant v_0 - v_1 \tag{7}$$

and define $v^* = bu_1 + (1 - b) v_1$. Then $u_1 < v^* \leqslant v_1$ and $Tv^* - v^* \leqslant 0$.[1]

[1] The relation $g < h$ denotes that $h - g \in K$, with $h \neq g$. For further terminology and notation about spaces with a cone see [3, 7].

Proof. Utilizing property (iii) we have

$$\begin{aligned} Tv^* - v^* &= f + B(bu_1 + (1-b)\,v_1\,, bu_1 + (1-b)\,v_1) - (bu_1 + (1-b)\,v_1) \\ &\leqslant B(bu_1 + (1-b)\,v_1\,, bu_1 + (1-b)\,v_1) \\ &\quad - B(bu_0 + (1-b)\,v_0\,, bu_0 + (1-b)\,v_0). \end{aligned}$$

From (7) and (i) it follows that the last expression is nonpositive. Q.E.D.

Let us note that, for $a > 0$ and $b > 0$, $u_2 < u^*$, $v^* < v_1$, i.e. the constructed elements lie closer to the solution than u_2, v_1.

In conclusion, whenever condition (3) on f is satisfied, the procedure can be described as follows.

Step 1. Set $u_0 = f$; choose c satisfying (4) and set $v_0 = (1 + c)f$.

Step 2. Iterate twice starting from u_0 to obtain u_1, u_2 and once from v_0 to obtain v_1.

Step 3. Construct u^*, v^* as indicated in Theorems 2 and 3 respectively.

Step 4. Set $u_0 = u^*$, $v_0 = v^*$ and go to Step 2.

The process can be stopped at any stage when the convergence criterion is met.

IV. Numerical Results

Rall [10] presents a detailed investigation of the problems arising during implementation of the method of monotone approximations on an automatic computing machine and describes an algorithm based upon the construction of lower and upper discrete bounding operators for T; an analogous discussion can also be found in [5].

Here we consider the equation [6, p. 141]

$$z(x) = \tfrac{1}{4}(1 - x^2) + z(x) \int_0^1 \frac{x}{x + t}\, z(t)\, dt, \tag{8}$$

to illustrate the accuracy and speed of the given acceleration technique. Actually, we have $\|f\| = \frac{1}{6}$, $M = \frac{1}{8}$ so that $3 - 2\sqrt{2} \leqslant c \leqslant 3$. We put $c = \frac{1}{3}$. Then $u_0 = f$, $v_0 = \frac{1}{3}(1 - x^2)$.

From the definition of u^* and v^* it is seen that, in actual computations, the constants a, b should be selected as large as possible, in order to get an improved estimate of z^*; conditions (6.1), (6.2), and (7) ensure that they do not become "too large." Moreover, we note that condition (6.1) can be rewritten as

$$a \leqslant \tfrac{1}{2}(1 - \|u_1\|^2)/(\|f\| + \tfrac{1}{2}\|u_1\|^2 - \|u_1\|)$$

because $\| u_2 - u_1 \| = \| u_2 \| - \| u_1 \|$ and $\| u_2 \| = \| f \| + \frac{1}{2} \| u_1 \|^2$ [3]. Therefore the explicit computation of $\| u_2 \|$ is not needed. $\| u_1 \|$ can be evaluated with no additional effort during the iteration which gives u_2.

As the results show, the error is less than 0.5×10^{-4} after six iterations (four from below and two from above) using acceleration (Table I), while the same performance is obtained after 11 iterations without acceleration (Table II). All results were obtained on the IBM 370/165 of the CNEN Computer Center using single precision arithmetic.

TABLE I

Approximate Solution of Equation (8) Using Acceleration

t	u_0	u_1^*	u_2^*	v_2^*	v_1^*	v_0
0	.25000	.25000	.25000	.25000	.25000	.33333
.1	.24750	.26123	.26138	.26140	.26269	.33000
.2	.24000	.25967	.25999	.26002	.26159	.32000
.3	.22750	.25021	.25067	.25070	.25230	.30333
.4	.21000	.23374	.23429	.23432	.23583	.28000
.5	.18750	.21062	.21122	.21125	.21260	.25000
.6	.16000	.18105	.18164	.18167	.18281	.21333
.7	.12750	.14515	.14567	.14569	.14659	.17000
.8	.09000	.10297	.10338	.10339	.10401	.12000
.9	.047500	.054579	.054812	.054818	.055138	.063333
1.	0	0	0	0	0	0

TABLE II

Approximate Solution of Equation (8) Using Sequences (2) and (5)

t	u_0	u_4	u_5	v_6	v_2	v_0
0	.25000	.25000	.25000	.25000	.25000	.33333
.1	.24750	.26138	.26138	.26139	.26230	.33000
.2	.24000	.25998	.25999	.26000	.26156	.32000
.3	.22750	.25065	.25067	.25068	.25264	.30333
.4	.21000	.23427	.23429	.23431	.23646	.28000
.5	.18750	.21119	.21122	.21124	.21340	.25000
.6	.16000	.18162	.18164	.18166	.18368	.21333
.7	.12750	.14565	.14567	.14569	.14741	.17000
.8	.09000	.10336	.10338	.10339	.10467	.12000
.9	.047500	.054801	.054810	.054818	.055525	.063333
1.	0	0	0	0	0	0

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